A full formulation-based soil-water coupled finite deformation analysis on undrained compression tests on highly permeable soil specimen

L'analyse couplée sol-eau de déformation finie basée sur une formulation complète d'une compression sans drain d'échantillons de sols très perméables

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ABSTRACT: In the mechanics of saturated soil based on the theory of mixtures, the pore water pressure *p* and the displacements ***u*** and ***U*** of the solid and liquid phases, respectively, are assumed to be independent variables that are unknowns of the field when solving the equation of motion of each phase by coupling with the law of conservation of mass. This is called full formulation. In many analysis codes for saturated soil, this strict formulation is modified to reduce the number of equations by assuming that the acceleration of the liquid phase relative to the solid phase is much smaller than that of the solid phase (***u***-*p* formulation). As this assumption neglects the dynamic permeation of pore water, it has not been possible to apply ***u***-*p* formulation to soils with very high permeability. This paper demonstrates that by utilizing the finite element method, the finite volume method, etc., it is possible to carry out deformation analysis of soils with very high permeability by solving the strict governing equations based on full formulation for finite deformation fields of soil-water coupled systems.

RÉSUMÉ : Dans la mécanique des sols saturés basée sur la théorie des mélanges, on considère que la pression interstitielle de l'eau *p* et les déplacements respectifs ***u*** et ***U*** des phases solide et liquide sont des variables indépendantes qui sont des inconnues du champ quand on résout l'équation de mouvement de chaque phase par couplage avec la loi de conservation de la masse. C'est ce que l'on appelle formulation complète. Dans de nombreux codes d'analyse de sols saturés, cette formulation stricte est modifiée pour réduire le nombre d'équations en considérant que l'accélération de la phase liquide par rapport à la phase solide est beaucoup plus petite que celle de la phase solide (formulation ***u***-*p*). Comme cette assomption la perméation dynamique de l'eau interstitielle, il n'a pas été possible d'appliquer la formulation ***u***-*p* aux sols à très haute perméabilité. Cet article démontre qu'en utilisant la méthode des éléments finis, la méthode des volumes finis, etc., il est possible d'effectuer une analyse de déformation de sols à très haute perméabilité en résolvant les équations principales strictes basées sur la formulation complète pour les champs de déformation finie de systèmes couplés sol-eau.

KEYWORDS: full formulation, soil-water coupled finite deformation analysis, theory of mixtures

1 INTRODUCTION

The dramatic improvement in computer performance in recent years is making it possible to solve problems at reasonable speeds even if there are a great number of equations. Against this background, it goes without saying that “massive-scale computation,” in which the calculation methods are applied to problems with larger numbers of elements, is also becoming possible. On another front, it has also become possible now to carry out “high-accuracy computations” to obtain exact solutions of governing equations without resorting to the use of assumptions or approximations that had to be made in calculations up to now. One such example is finite deformation theory. It allows large deformation analysis to be performed with consideration of the effects of geometric nonlinearity, i.e., shape variation, which is disregarded in the case of infinitesimal deformation theory.

The mechanics of saturated soil are based on the theory of mixtures, and in most cases, the displacement ***u*** of the soil skeleton and the pore water pressure *p* are taken as independent variables for soil-water coupled analyses utilizing ***u***-*p* formulation. This method reduces the number of equations handled by assuming that the acceleration of the liquid phase relative to the solid phase is much smaller than that of the solid phase, thus making it possible to save computational resources significantly. However, the price paid for this simplification is that since the method disregards the phenomenon of dynamic permeation of pore water, it cannot be used for analyzing cases where there is dynamic movement of the pore water in grounds with very high permeability. In contrast, full formulation (***u***-***U***-*p* formulation or ***u***-***w***-*p* formulation) (Biot 1956) makes it possible to solve the strict system of equations without employing the assumption mentioned above. The dependent variables in this case include the displacement ***U*** (or relative displacement ***w***) of the pore water in addition to the displacement ***u*** of the soil skeleton and the pore water pressure *p*. Zienkiewicz et al. (1999) have carried out infinitesimal deformation linear elastic analysis based on full formulation (***u***-***w***-*p* formulation) and have shown that the solutions obtained through ***u***-*p* formulation differ substantially from those obtained through full formulation in the case of problems where large coefficients of permeability or high frequencies are encountered. It can therefore be expected that by using full formulation and taking account of the dynamic response of pore water, it would be possible to carry out analysis of materials with high permeability. Hitherto, it has not been possible under such situations to obtain solutions through elasto-plastic finite deformation analysis based on ***u***-*p* formulation. This paper first outlines a newly developed method of analysis based on full formulation, comparing it with the dynamic/static soil-water coupled finite deformation analysis code ***GEOASIA*** (Noda et al. 2008) used to solve rate-type equations of motion based on ***u***-*p* formulation. In addition, it is demonstrated that the new method is capable of analyzing materials with very high permeability.

2 OUTLINE OF GOVERNING EQUATIONS BASED ON FULL FORMULATION

Based on the theory of mixtures, let us consider solid and liquid phases that are the respective continuous bodies equivalent to soil skeleton and pore water, which are the constituent elements of saturated soil. In soil-water coupled analysis, the soil skeleton deformation phenomenon and the water permeation phenomenon are coupled when solving the system of equations. As a result, when basing the analysis on full formulation (***u***-***U***-*p* formulation), the three unknowns in the field would be the velocity $\boldsymbol{v}_s$ of the solid phase, velocity $\boldsymbol{v}_f$ of the liquid phase, and pore water pressure u. The three governing equations below are used to determine these unknowns. The updated Lagrange method is used to take account of the geometric nonlinearity of the material. Consequently the equation of motion applied is a rate-type one, and a jerk term appears in it.

Rate-type equation of motion of mixtures:

$$\rho_s D_s^2 \boldsymbol{v}_s + \rho_f D_s D_f \boldsymbol{v}_f + \rho^f (\mathrm{tr}\boldsymbol{D}_s)(D_f \boldsymbol{v}_f - \boldsymbol{b}) = \mathrm{div}(D_s \boldsymbol{S}_t) \quad (1)$$

Equation of motion of the liquid phase (Darcy's law with consideration for acceleration):

$$\rho^f D_f \boldsymbol{v}_f = -\gamma_w \mathrm{grad}\ h - \frac{\gamma_w}{k} n(\boldsymbol{v}_f - \boldsymbol{v}_s) \quad (2)$$

Soil-water coupled equation (Law of conservation of mass):

$$\frac{\rho^f k}{\gamma_w} \mathrm{div}(D_f \boldsymbol{v}_f) - \mathrm{div}\boldsymbol{v}_s + \mathrm{div}(k\,\mathrm{grad}h) = 0 \quad (3)$$

In the above equations, ρ, ρ_s, ρ_f and ρ^f, are the densities of the mixture, solid phase, liquid phase, and pore water, respectively. D_s and D_f denote the material derivative operators as seen from the solid phase and liquid phase, respectively. $\boldsymbol{D}_s$ is the velocity gradient tensor, n is the pore ratio, and K_f is the bulk elastic modulus of pore water. The body force is indicated by $\boldsymbol{b}=-g\boldsymbol{e}_3$, where g is the acceleration due to gravity and $\boldsymbol{e}_3$ is the vertical upward unit vector. $D_s\boldsymbol{S}_t$ is the nominal stress rate in conformity with the method of notation of Yatomi et al. (1989), γ_w is the unit weight of pore water, h is the total water head, and k is the coefficient of permeability. For simplicity, compression of water is omitted in the above equations.

In the case of ***u***-*p* formulation, the full formulation governing equations (1) and (3) above are simplified by introducing the assumption that the acceleration of the pore water relative to the soil skeleton is much smaller than that of the soil skeleton ($D_s\boldsymbol{v}_s \gg D_f\boldsymbol{v}_f - D_s\boldsymbol{v}_s$), which in effect deletes the liquid phase velocity $\boldsymbol{v}_f$ from the equations. Therefore, the governing equations based on ***u***-*p* formulation take the forms shown below.

Rate type equation of motion of mixtures:

$$\rho D_s^2 \boldsymbol{v}_s + \rho^f (\mathrm{tr}\boldsymbol{D}_s)(D_s \boldsymbol{v}_s - \boldsymbol{b}) = \mathrm{div}(D_s \boldsymbol{S}_t) \quad (4)$$

Soil-water coupled equation (Law of conservation of mass):

$$\frac{\rho^f k}{\gamma_w} \mathrm{div}(D_s \boldsymbol{v}_s) - \mathrm{div}\boldsymbol{v}_s + \mathrm{div}(k\,\mathrm{grad}h) = 0 \quad (5)$$

Since the velocity $\boldsymbol{v}_f$ of the liquid phase is not explicitly sought in the method of analysis employing ***u***-*p* formulation, there is no necessity of solving the equation of motion (2) of the liquid phase, and only equations (4) and (5) are coupled and solved. In the method of analysis employing full formulation, equations (1) to (3) are coupled and solved strictly to include the motion of the liquid phase too.

For the above governing equations, spatial discretization of the motion of the solid and liquid phases is carried out by the finite element method using 4-node isoparametric elements. A finite volume method, which employs the extended Christian and Tamura's physical model that assigns the value at the center of each element, is used for the pore water pressure (Christian 1968, Akai and Tamura 1978, Asaoka et al. 1994). Time integration for the motion of the solid and liquid phases is performed by Wilson's θ method that assumes linearity of the "jerk" term, which is the material time-derivative that focuses on the acceleration of the solid phase. Time integration for the pore water pressure and other quantities of state is performed by the trapezoidal rule. Details of time and spatial discretization have been omitted here because of space limitations. Although equations of motion do not possess objectivity, the Green-Naghdi(1965)'s rate is used as the stress rate with objectivity in the constitutive equations. The constitutive equations are mounted with the SYS Cam-clay model (Asaoka et al. 2002), which is an elasto-plastic constitutive model that focuses on the soil skeleton structure. The third term on the right side of equation (2) is the force of interaction (the body force equivalent to the seepage force) of the solid phase on the liquid phase. It is obtained by analogy to Hagen-Poiseuille flow using Nishimura(1999)'s work as a reference.

3 SIMULATION OF UNDRAINED COMPRESSION TESTS ON LOW PERMEABILITY SPECIMENS (VERIFICATION OF THE ABOVE METHOD OF ANALYSIS)

The deformation problem solved here using the analysis code based on the method developed is for a plane strain specimen compressed in the vertical direction under constant lateral pressure and undrained conditions as shown in Fig 1. If the material is impermeable (k=0), there will be no migration of pore water within the specimen, which means that the solid and liquid phases will move together ($\boldsymbol{v}_s=\boldsymbol{v}_f$). In this section, a low permeability material (k=10^{-7}cm/s) and one having high permeability (k=10^{3}cm/s) were envisaged, and analysis was carried out under conditions that allow migration of pore water.

3 .1 *Analysis conditions*

The initial conditions, boundary conditions, and material constants are described below.

Initial conditions: The displacement rates $\boldsymbol{v}_s$ and $\boldsymbol{v}_f$ of the solid and liquid phases were made zero assuming that the specimens are at rest initially, and the lateral pressure was assumed to be 300 kPa.

Boundary conditions of the soil skeleton: Specimens 10 cm × 10 cm in size were compressed under plane strain conditions with the upper and lower ends being displaced at an axial displacement rate of 10^3 m/s. Although not realistic, this extremely high displacement rate makes manifestation of the dynamic effects of the soil skeleton and pore water possible. The upper and lower ends were assumed to be rigid and frictionless. Even under such a frictionless state, this analytical method, which takes account of inertia, makes it possible for

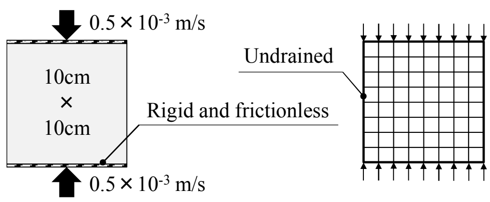

Figure 1. Undrained plane strain compression problem

the specimen to reach a state that allows migration of pore water to occur as loading progresses without exhibiting the phenomenon of uniform deformation in which the rectangular shape remains intact (Noda et al. 2013).

Hydraulic boundary conditions: Undrained tests were simulated assuming that all four boundary faces were in the undrained condition. In addition, the displacement rate of the liquid phase in the direction perpendicular to the loading surface was made the same as the displacement rate of the solid phase.

Material constants and initial conditions: The material constants and initial conditions of the SYS cam-clay model were set as in Table 1, assuming normally consolidated remolded soil in the condition of isotropic stress. As for the coefficients of permeability, two types of materials were assumed, one having low permeability (k=10^{-7}cm/s) and the other having high permeability (k=10^{3}cm/s). The time interval width was set at Δt=10^{-5} s.

3 .2 *Analysis results of the low permeability specimen*

The distributions of the relative velocity of pore water, pore water pressure, and specific volume obtained through the analysis based on full formulation are shown in Fig 2, Fig 3, and Fig 4, respectively. It can be seen from Fig 2 that almost no migration of water occurred within the specimen. This is because the coefficient of water permeability of the material is extremely small. Corresponding to this, the specific volume distribution within the specimen is nearly uniform, as seen from Fig 4, and does not vary with the passage of time. This is because there is no movement of pore water between the elements and almost no volumetric change in any soil element. In contrast, the pore water pressure (Fig 3) is distributed nonuniformly within the specimen. This is due to the effect of the inertial term, which causes the top and bottom end of the specimen to start deforming before the center of the specimen does, thus resulting in nonuniform deformation of the specimen. Although omitted here because of space restrictions, it was confirmed that if the analysis was carried out under the same conditions using ***u***-*p* formulation (***GEOASIA***), the results

Table 1. Material constants of the SYS cam-clay model

Elasto-plastic parameters	
Compression index $\tilde{\lambda}$	0.500
Swelling index $\tilde{\kappa}$	0.040
NCL intercept N	2.40
Critical state index M	1.40
Poisson's ratio ν	0.30
Physical properties	
Soil particle density ρ^s(g/cm^3)	2.65
Water density ρ^f(g/cm^3)	1.00
Coefficient of permeability k(cm/s)	10^{-7} or 10^3
Unit weight of water γ_w(kN/m^3)	9.81
Initial conditions	
Overconsolidation ratio $1/R_0$	1.0
Degree of structure $1/R_0^*$	1.0
Stress ratio η_0	0.0
Degree of anisotropy ζ_0	0.0
Specific volume v_0	1.598

would be almost exactly the same as those described above. This indicates that the assumptions made in ***u***-*p* formulation are sufficiently valid when the coefficient of water permeability is small enough to make the movement of pore water relative to the soil skeleton almost static. To put it differently, the above means that the analysis code developed here has been verified.

3 .3 *Analysis results of the high permeability specimen*

The transitions of the relative velocity of pore water, pore water pressure, and specific volume obtained through the analysis based on full formulation are shown in Fig 5, Fig 6, and Fig 7, respectively. It can be observed from Fig 5 that dynamic migration of pore water occurred within the specimen. This is due to the extremely high water permeability. As is evident from Figure 6, in such a state, the pore water pressure is always uniformly distributed within the element. This is due to the quick elimination of the hydraulic gradient caused by the immediate migration of water. On the other hand, the specific volume distribution (Fig 7) varies widely with elapse of time. This is caused by the dynamic movement of pore water through the specimen, which results in repeated drainage/water absorption and contraction/expansion occurring in each soil

(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 2. Relative velocity distributions of pore water (k=10^{-7}cm/s)

(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 3. Pore water pressure distributions (k=10^{-7}cm/s)

(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 4. Specific volume distributions (k=10^{-7}cm/s)

element even though the volume of the specimen as a whole may remain constant. It was not possible to carry out this computation using ***u***-*p* formulation. The reason for this is that if the water permeability is high, the assigned conditions are such that the discretized soil-water coupled equation of motion does not comply with the law of mass conservation, which requires water absorption/drainage to occur in correspondence with the compression/expansion of the soil skeleton.

4 CONCLUSION

A method of soil-water coupled elasto-plastic finite deformation analysis based on full formulation was developed in this study. It has been demonstrated that the developed method is capable of two-phase analysis of materials possessing extremely high water permeability, which cannot be performed using ***u***-*p* formulation. It was also shown that full formulation, which seeks directly the velocity of the liquid phase, makes direct observation of migration of pore water in soil possible. In the future, the authors will attempt to elucidate problems related to the dynamic interactions between pore water and soil skeleton, including the dynamic behavior of grounds during an earthquake, behavior of soil structures due to tsunami collision/scouring, and other phenomena such as riverbed disturbance and seabed liquefaction.

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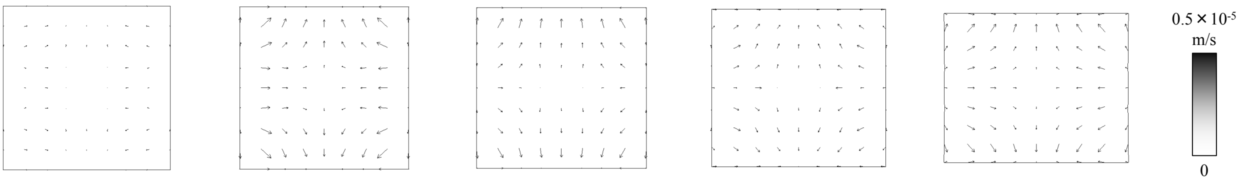
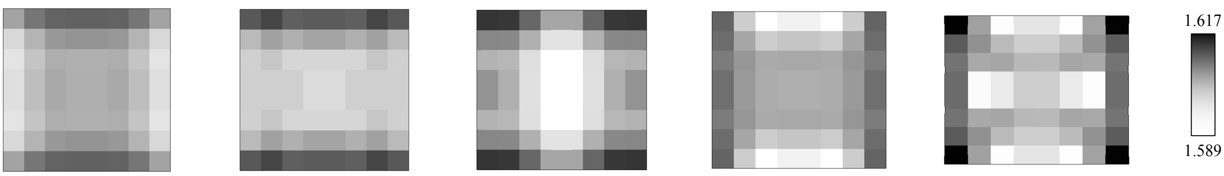


(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 5. Relative velocity distributions of pore water (k=10^{3}cm/s)

(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 6. Pore water pressure distributions (k=10^{3}cm/s)



(a) ε_a=1% (b) ε_a=2% (c) ε_a=3% (d) ε_a=5% (e) ε_a=10%

Figure 7. Specific volume distributions (k=10^{3}cm/s)